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THE PLANE PROBLEM OF THE THEORY OF CONVECTIVE HEAT AND MASS TRANSFER

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An effective solution is obtained for the plane problem of convective heat transfer from a heated body of arbitrary shape in a stream of perfect incompressible heat-conducting fluid. Solution of this problem is of interest in the theory of heat transfer in liquid metals, of the mass transfer of a bubble moving in a fluidized bed, in the theory of ablation and freezing in a stream of heat-conducting fluid, etc. The input boundary value problem in Helmholtz' variables reduces to the similar problem of convective heat transfer from a heated plate in a longitudinal stream of perfect incompressible fluid, a problem that is solved, after some transformations, by separating variables in elliptic coordinates. The solution is in the form of series in Mathieu's functions. Simple asymptotic formulas are obtained for low and high Péclet numbers. A simple interpolation formula is obtained for the heat flux over a range of Péclet numbers.

1. Statement of the problem. We consider a heated cylinder whose generatrix is parallel to the z-axis so that its cross section contour lies in the xy-plane. The cylinder is in a steady irrotational plane-parallel flow of perfect incompressible heat-conducting fluid with thermal diffusivity a and thermal conductivity k_f. The fluid velocity at infinity is v_0 and is directed along the x axis. The cylinder temperature T_0 is assumed constant and that of the fluid at infinity to be zero. We have to determine the temperature field and the heat flux from the body to the fluid.

The assumption of perfect fluid with its associated smallness of the viscous boundary layer thickness in comparison with the characteristic dimension of the cross section (or with the thickness of the thermal boundary layer at high Péclet numbers) is justified, for instance, for molten metals used as coolants in atomic reactors [1].

This problem is also of interest in the theory of mass transfer in the case of a bubble moving in a fluidized bed [2], when the Péclet number, as determined by a linear dimension of the closed circulation region, the bubble relative velocity, and by the effective diffusion coefficient, can assume widely varying values.

The method of freezing is used in mining for cutting wells and shafts in formation containing underground water. The problem is further encountered in the calculation of thickness of ice-forming bodies and of heat fluxes [3]. Other fields of its possible technological applications also exist.

The boundary value problem is of the form

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a\Delta T \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.1}$$

$$v_x = v_0 \frac{\partial \varphi}{\partial x}, \quad v_y = v_0 \frac{\partial \varphi}{\partial y}, \quad \Delta\varphi = 0$$
$$T = 0, \quad v_x = v_0, \quad v_y = 0, \quad x^2 + y^2 \to \infty \qquad (1.2)$$
$$T = T_0, \quad \partial\varphi / \partial n = 0 \text{ on } L$$

where T is the temperature, v_x and v_y are fluid velocity components, and n is the direction of a normal to the contour. The potential and the stream function of the fluid potential flow (of dimension of length) are denoted by φ and ψ, respectively. Since for a given contour φ and ψ are some functions of x and y, they can be determined by methods of the theory of functions of complex variable. We

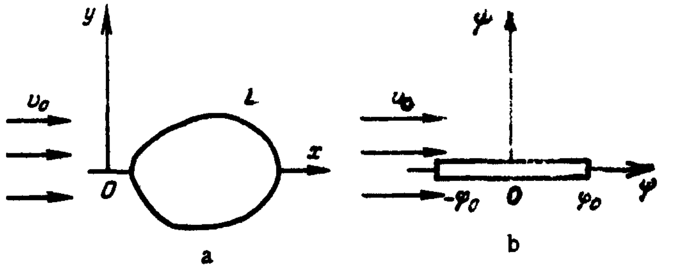

Fig. 1

assume these functions to be known. It can be assumed without loss of generality that at the contour L $\psi = 0$, $-\varphi_0 < \varphi < \varphi_0$. . Passing in formulas (1.1) to variables φ and ψ, we obtain the following boundary value problem (see Fig. 1):

$$v_0 \frac{\partial T}{\partial \varphi} = a \left(\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial \psi^2}\right) \qquad (1.3)$$
$$T = T_0, \quad \psi = 0, \quad |\varphi| < \varphi_0; \quad T = 0, \ \varphi^2 + \psi^2 \to \infty$$

In variables $\varphi\psi$ we thus obtain the problem of convective heat transfer form a heated plate of width $2\varphi_0$ in a lengthwise stream of heat-conducting perfect fluid flowing at velocity v_0 (Fig. 1).

We introduce the new function

$$u = Te^{-\lambda\varphi} \quad (\lambda = v_0 / (2a)) \qquad (1.4)$$

and obtain the following boundary value problem:

$$\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \psi^2} = \lambda^2 u \qquad (1.5)$$
$$u = T_0 e^{-\lambda\varphi}, \quad \psi = 0, \quad |\varphi| < \varphi_0$$

2. Solution of the boundary value problem. We pass to elliptical coordinates ξ and η

$$\varphi = \varphi_0 \operatorname{ch}\xi \cos\eta, \quad \psi = \varphi_0 \operatorname{sh}\xi \sin\eta \tag{2.1}$$

and, setting $u = Z(\xi)\Psi(\eta)$, separate variables. For $\Psi(\eta)$ we obtain the Mathieu equation and for $Z(\xi)$ the modified Mathieu equation

$$\partial^2\Psi / \partial\eta^2 + (h - 2q\cos 2\eta)\Psi = 0 \tag{2.2}$$

$$\frac{\partial^2 Z}{\partial \xi^2} - (h - 2q \operatorname{ch} 2\xi) Z = 0, \quad q = -k^2 = -\left(\frac{v_0\varphi_0}{4a}\right)^2 < 0 \tag{2.3}$$

where h is the separation constant and k is the Péclet number.

Conditions of the problem clearly imply that solution (2.2) must be even and periodic of period π or 2π. Mathieu functions of the form $\mathrm{ce}_{2m}(\eta, -q)$ and $\mathrm{ce}_{2m+1}(\eta, -q)$, where $m = 0, 1, \ldots$ are such solutions of Eq. (2.2). Eigenvalues $h = a_{2m}(q)$ and $h = b_{2m+1}(q)$ correspond to these functions.

The solutions of Eq. (2.3) must be real and vanish when $\xi \to \infty$. The modified Mathieu functions $\mathrm{Fek}_{2m}(\xi, -q)$ and $\mathrm{Fek}_{2m+1}(\xi, -q)$ are such solutions.

All notation related to Mathieu functions conforms to that in [4, 5], and subscripts $2m$ and $2m+1$ indicate that when $q = 0$

$$\mathrm{ce}_{2m}(\eta, 0) = \cos 2m\eta, \quad \mathrm{ce}_{2m+1}(\eta, 0) = \cos(2m+1)\eta$$

The general solution of Eq. (1.5) is of the form

$$u = \sum_{n=0}^{\infty} \alpha_n \mathrm{ce}_n(\eta, -q)\,\mathrm{Fek}_n(\xi, -q) \tag{2.4}$$

Functions $\mathrm{ce}_n(\eta)$ constitute a complete orthogonal system, hence the boundary condition can be expanded in series in $\mathrm{ce}_n(\eta, -q)$ when $\xi = 0$

$$e^{-2k\cos\eta} = \sum_{n=0}^{\infty} f_n \mathrm{ce}_n(\eta, -q) \tag{2.5}$$

$$f_n = \frac{1}{\pi}\int_0^{2\pi} \mathrm{ce}_n(\eta, -q)\, e^{-2k\cos\eta}\, d\eta$$

For the determination of f_n we use formulas [5]

$$\mathrm{ce}_{2m}(\eta, -q) = (-1)^m \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} \cos 2r\eta \tag{2.6}$$

$$\mathrm{ce}_{2m+1}(\eta, -q) = (-1)^m \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2m+1)} \cos(2r+1)\eta$$

where $A_{2r}{}^{(2m)}$, and $B_{2r+1}^{(2m+1)}$ are coefficients of the expansion of Mathieu functions in series whose numerical values are tabulated in [4, 6].

Since a term by term integration of such series is possible, we have

$$f_{2m} = (-1)^m 2 \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} I_{2r}(-2k) \tag{2.7}$$

$$f_{2m+1} = (-1)^m 2 \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2m+1)} I_{2r+1}(-2k)$$

We recall the following representation [4]:

$$\int_0^{2\pi} \cos m\eta e^{t \cos \eta} d\eta = 2\pi I_m(t) \tag{2.8}$$

where $I_m(t)$ are modified m-th order Bessel functions of the first kind.

Thus the solution of the boundary value problem is of the form

$$T(\eta, \xi) = 2T_0 e^{2k \cos \eta \operatorname{ch} \xi} \sum_{n=0}^{\infty} D_n \mathrm{ce}_n(\eta, -q) \mathrm{Fk}_n(\xi, -q) \tag{2.9}$$

$$\mathrm{Fk}_n(\xi, -q) = \frac{\mathrm{Fek}_n(\xi, -q)}{\mathrm{Fek}_n(0, -q)}$$

$$D_{2m} = (-1)^m \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} I_{2r}(2k) = \frac{\mathrm{ce}_{2m}(0, -q)}{\mathrm{ce}_{2m}(0, q)} A_0^{(2m)}$$

$$D_{2m+1} = (-1)^{m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2m+1)} I_{2r+1}(2k) = -\frac{\mathrm{ce}_{2m+1}(0, -q)}{\mathrm{ce}'_{2m+1}(0, q)} k B_1^{(2m+1)}$$

Representation of D_n in terms of Mathieu functions of zero argument makes possible the use of computation tables (e.g., in [6]).

Let us determine the heat flux Q from the body to the fluid (per unit of cylinder length)

$$Q = -k_f \oint_L \mathrm{grad}_n T\, dl = -2k_f \int_{-\varphi_0}^{\varphi_0} \frac{\partial T}{\partial \psi}\Big|_{\psi=0} d\varphi = -2k_f \int_0^{\pi} \frac{\partial T}{\partial \xi}\Big|_{\xi=0} d\eta \tag{2.10}$$

From solution (2.9) we obtain that $\partial T / \partial \xi$ for $\xi = 0$. Integrating term by term and applying integrals (2.8), from (2.10) we obtain the formula

$$Q^* = \frac{Q}{4\pi k_f T_0} = \sum_{n=0}^{\infty} (-1)^{n+1} D_n^2 \, \mathrm{Fk}_n'(0, -q) \tag{2.11}$$

$$\mathrm{Fk}_n'(\xi, -q) = \frac{\partial}{\partial \xi} \mathrm{Fk}_n(\xi, -q) = \frac{\mathrm{Fek}_n'(\xi, -q)}{\mathrm{Fek}_n(0, -q)}$$

$$\pi [A_0^{(2m)}]^2 \, \mathrm{Fek}_{2m}'(0, -q) = (-1)^{m+1} \mathrm{ce}_{2m}(\pi/2, q)\, \mathrm{ce}_{2m}^2(0, q)$$

$$\pi k^2 [B_1^{(2m+1)}]^2 \, \mathrm{Fek}_{2m+1}'(0, -q) = (-1)^{m+1} \mathrm{ce}_{2m+1}(\pi/2, q)\, [\mathrm{ce}_{2m+1}'(0, q)]^2$$

3. Analysis of solution. Low Péclet numbers. We investigate here the behavior of formulas (2.9) and (2.11) at small $|q|$, which corresponds to low Péclet numbers k. All functions appearing in the solution can be represented by series in trigonometric or Bessel functions with coefficients A_{2r} or B_{2r+1}. For this it is necessary to determine the behavior of the latter for $k \to 0$. This is generally achieved by using recurrent relationships and normalization conditions of the type (for $\mathrm{ce}_{2m}(\eta, q)$)

$$a_{2m} A_0 - q A_2 = 0, \quad (a_{2m} - 4) A_2 - q (A_4 + 2A_0) = 0 \tag{3.1}$$

$$(a_{2m} - 4r^2) A_{2r} - q (A_{2r+2} + A_{2r-2}) = 0$$

$$\frac{1}{\pi} \int_0^{2\pi} \mathrm{ce}_{2m}^2(\eta, q)\, d\eta = 2 [A_0^{(2m)}]^2 + \sum_{r=1}^{\infty} [A_{2r}^{(2m)}]^2 = 1$$

The principal terms of expansions of coefficients $A_{2r}^{(2m)}$ and $B_{2r}^{(2m+1)}$ at small k were obtained by Mathieu in the form [4]

$$A_{m+2p}^{(m)} \approx B_{m+2p}^{(m)} \approx (-1)^p \frac{m!}{p!\,(m+p)!} \left(\frac{k}{2}\right)^{2p} \tag{3.2}$$

$$A_{m-2p}^{(m)} \approx B_{m-2p}^{(m)} \approx \frac{(m-p-1)!}{p!\,(m-1)!} \left(\frac{k}{2}\right)^{2p}, \quad m > 0 \quad p \geqslant 0$$

According to these estimates coefficients $A_m^{(m)}$ and $B_m^{(m)}$ are of order unity. A more exact expansion of these coefficients is of the form

$$A_m^{(m)} \approx 1 - E_m (k/2)^4$$

where E_m are coefficients yet to be determined.

Using (3.2) and the normalization condition (3.1), from the last formula we obtain

$$A_1^{(1)} \approx 1 - \frac{1}{8}\left(\frac{k}{2}\right)^4, \quad A_2^{(2)} \approx 1 - \frac{19}{18}\left(\frac{k}{2}\right)^4$$
$$A_m^{(m)} \approx 1 - \frac{m^2+1}{(m^2-1)^2}\left(\frac{k}{2}\right)^4, \quad m > 2 \tag{3.3}$$

Expansions of coefficients $A_{2r}{}^{(0)}$ are similarly derived

$$A_0^{(0)} \approx \frac{\sqrt{2}}{2}\left[1 - \left(\frac{k}{2}\right)^4 + \frac{19}{4}\left(\frac{k}{2}\right)^8\right]$$
$$A_{2r}^{(0)} \approx (-1)^r \frac{\sqrt{2}}{(r!)^2}\left(\frac{k}{2}\right)^{2r}\left[1 - \frac{4r^2+6r+1}{(r+1)^2}\left(\frac{k}{2}\right)^4\right], \quad r \geqslant 1 \tag{3.4}$$

Using (3.3) and (3.4), from (2.6) we obtain expansions for $\mathrm{ce}_0(\eta, \pm q)$ in powers of k^2 (or $|q|$). For example

$$\mathrm{ce}_0(\eta, -q) \approx \frac{\sqrt{2}}{2}\left[1 + 2\left(\frac{k}{2}\right)^2 \cos 2\eta - \left(\frac{k}{4}\right)^4\left(1 - \frac{\cos 4\eta}{2}\right)\right]$$

Using the representations [4]

$$\mathrm{Fek}_{2m}(\xi, -q) = (-1)^m \frac{\mathrm{ce}_{2m}(0, q)}{\pi A_0^{(2m)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} K_{2r}(2k \operatorname{ch} \xi) \tag{3.5}$$
$$\mathrm{Fek}_{2m+1}(\xi, -q) =$$
$$(-1)^m \frac{\mathrm{se}'_{2m+1}(0, q)}{\pi k B_1^{(2m+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2m+1)} K_{2r+1}(2k \operatorname{ch} \xi)$$

and expansions of modified Bessel functions K_n of the third kind in powers of the argument, and formulas (3.3) and (3.4), we obtain expansions for $\mathrm{Fek}_n(\xi, -q)$.

As the result we have the following formulas:

$$\frac{T}{T_0} \approx \left\{1 + \frac{\ln \operatorname{ch} \xi + \varkappa_0(0) - \varkappa_0(\xi)}{\ln \gamma k} - \frac{4 \cos \eta \,[1 + \varkappa_1(\xi)]}{\operatorname{ch} \xi \,[1 + \varkappa_1(0)]} k\right\} \times$$
$$\exp(2k \cos \eta \operatorname{ch} \xi) \quad (k \ll 1) \tag{3.6}$$

$$Q^* \approx -(1 + 9k^2/4)\ln^{-1}(\gamma k/2) - 2k^2 \tag{3.7}$$
$$\varkappa_0(\xi) = \sum_{i=1}^{\infty} g_i, \quad \varkappa_1(\xi) = \sum_{i=1}^{\infty} \frac{2i}{i+1} g_i, \quad g_i = \frac{(2i-1)!}{4^i (i!)^2 \operatorname{ch}^{2i} \xi}$$

where $\ln\gamma = c$, $c = 0.577...$ is the Euler's constant.

High Péclet numbers. Physical considerations indicate clearly that at high k the fluid is heated only in a comparatively thin layer stretching in the direction of the φ-axis, i.e. at high k the condition

$$\partial^2 T / \partial\psi^2 \gg \partial^2 T / \partial\varphi^2$$

of the thermal boundary layer is satisfied.

It is thus possible to obtain asymptotic estimates at high Péclet numbers using the solution of the following self-similar problem of the thermal boundary layer:

$$v_0 \frac{\partial T}{\partial \varphi} = a \frac{\partial^2 T}{\partial \psi^2} \quad (\varphi > -\varphi_0,\ \psi > 0)$$
$$T = T_0, \quad \psi = 0; \quad T = 0, \quad \varphi = -\varphi_0$$

The solution of that problem is of the form

$$T = T_0 \operatorname{erfc}\left\{\left[\frac{k\psi^2}{\varphi_0(\varphi+\varphi_0)}\right]^{1/2}\right\} \tag{3.8}$$

$$Q^* = (2/\pi)^{1/2} \sqrt{k} (k \gg 1) \tag{3.9}$$

In the boundary layer approximation solution (3.8) is obviously not valid in the neighborhood of the singular point $\varphi = -\varphi_0$, $\psi = 0$, where it is necessary to use the exact general solution (2.9), (2.11).

Neighborhood of the singular point. The field near the singular point $\varphi = -\varphi_0$, $\psi = 0$ can be determined for any Péclet number using the solution of the following boundary value problem [7]:

$$\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \psi^2} = \lambda^2 u, \quad u = Te^{-\lambda\varphi} \tag{3.10}$$
$$u = T_0 e^{-\lambda\varphi} = u_0 e^{-\lambda(\varphi+\varphi_0)}, \quad \psi = 0, \ \varphi > -\varphi_0$$
$$\partial u / \partial\psi = 0, \quad \psi = 0, \ \varphi < -\varphi_0$$

We pass to polar coordinates $r\theta$

$$\varphi + \varphi_0 = r\cos\theta, \quad \psi = r\sin\theta$$

Using the method of separation of variables we obtain for the solution of (3.10) the following integral representation (of the Kontorovich—Lebedev type):

$$u = \int_0^\infty (A \operatorname{ch} \nu\theta + B \operatorname{sh} \nu\theta) K_{i\nu}(\lambda r)\, d\nu$$

in which A and B are unkown functions of ν, and which vanishes for $r \to \infty$.
Using the boundary conditions and the Kontorovich – Lebedev tables of integral transformations [9], we obtain

$$\pi u = 2u_0 \int_0^\infty (\operatorname{ch} \nu\theta - \operatorname{th} \nu\pi \operatorname{sh} \nu\theta) K_{i\nu}(\lambda r)\, d\nu$$

from which for the heat flux through segment $-\varphi_0 \leqslant \varphi \leqslant \varphi_0$ we have

$$Q = -2k_f \int_{-\varphi_0}^{\varphi_0} \frac{\partial T}{\partial \psi}\bigg|_{\psi=0} d\varphi = -2k_f e^{-\lambda\varphi_0} \int_0^{2\varphi_0} \frac{\partial u}{\partial \theta}\bigg|_{\theta=0} e^{\lambda r} \frac{dr}{r} = \tag{3.11}$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}} k_f T_0 \int_0^{2\lambda\varphi_0} \frac{dx}{\sqrt{x}} = \frac{8}{\sqrt{\pi}} k_f T_0 \sqrt{\lambda\varphi_0}$$

where we used the following relationship [8]:

$$\int_0^\infty \nu \operatorname{th} \nu\pi K_{i\nu}(x)\, d\nu = \sqrt{\frac{\pi x}{2}}\, e^{-x}$$

As the result, we again have formula (3.9).

It is remarkable that the respective formulas obtained earlier with the use of the approximate boundary layer theory is exactly the same as that derived using the exact singular solution in the singular point neighborhood.

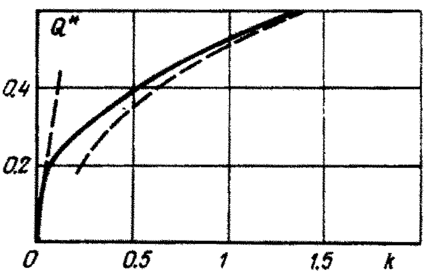

Fig. 2

4. Interpolation formula for the heat flux. The complex exact formula (2.11) for the heat flux is not convenient for calculations over the whole range of Péclet numbers; it can be approximated by the following simple formula:

$$Q_a^* = \frac{Hg^{M+1} + |G|^{-N}}{Hg^M + |G|^{-N-1}} \quad (Q^* \approx Q_a^*) \tag{4.1}$$

$$g = (2/\pi)^{1/2} \sqrt{k}, \quad G = -(1 + 9k^2/4) \ln^{-1}(\gamma k/2) - 2k^2$$

where H, M, and N are some constants. For $k \ll 1$ and $k \gg 1$ function $Q_a^*(k)$ behaves as the exact asymptotic solutions (3.7) and (3.9), respectively.
Constants H, M, and N are determined as follows. We calculate the mean

square interpolation error over ten points k_i uniformly spaced on the interval $0.01 < k < 5$

$$\sigma(H, M, N) = \left\{\frac{1}{9}\sum_{i=1}^{10}[Q^*(k_i) - Q_a^*(k_i)]^2\right\}^{1/2}$$

Quantities $Q^*(k_i)$ were obtained from the exact solution (2.11) using a computer. The nonlinear function $\sigma(H, M, N)$ was then minimized by Seidel's method of coordinate-wise descent. The following values of constants were obtained $H = 20$, $M = 0.02$, and $N = 1.30$. The error of interpolation by formula (4.1) did not exceed 2%.

Function $Q^*(k)$ is shown in Fig. 2 by the solid line calculated on a computer using formula (2.11). The asymptotic solutions (3.7) and (3.9) are shown there by dash lines. Thus solution (3.7) which is valid for approximate calculations for $k \leqslant 5 \cdot 10^{-3}$ yields results with an error of less than 4%, while for $k \geqslant 1$ formula (3.9) is valid, yielding results with an error of less than 2%.

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